Some properties of Laplace Transform

1. Linearity: Laplace Transform is a linear transform:

$$\mathcal{L}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 \mathcal{L}\{f_1\} + a_2 \mathcal{L}\{f_2\}$$

2. Time-delay:

$$\mathcal{L}\{f(t-T)\,u_s(t-T)\} = e^{-sT} F(s)$$

3. Differentiation:

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

4. Integration:

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s)$$

5. <u>Final value Theorem:</u>

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

only valid if all poles of $sF(s)$ are in the LHP.

Ex: $F(s) = \dfrac{5}{s(s^2+s+2)}$ $\qquad f(\infty) = ?$

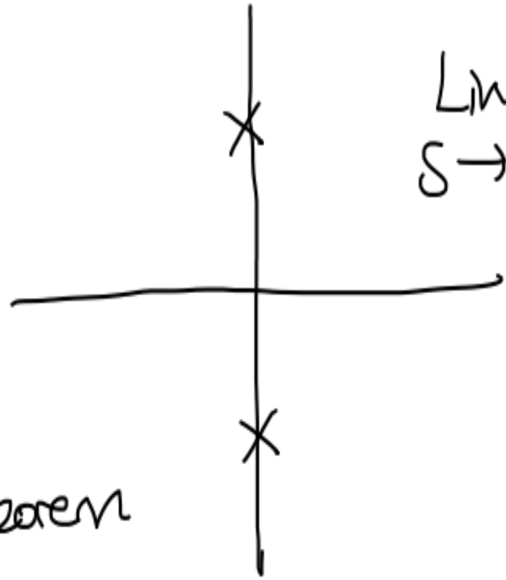
$sF(s)$ must be a stable TF ?

$$sF(s) = \cancel{s}\frac{5}{\cancel{s}(s^2+s+2)} = \frac{5}{s^2+s+2} \leftarrow \text{stable}$$

$$\Rightarrow f(\infty) = \lim_{s \to 0} sF(s) = \frac{5}{2}$$

EX: $F(s) = \dfrac{4}{s^2+4}$, $f(\infty) = ?$

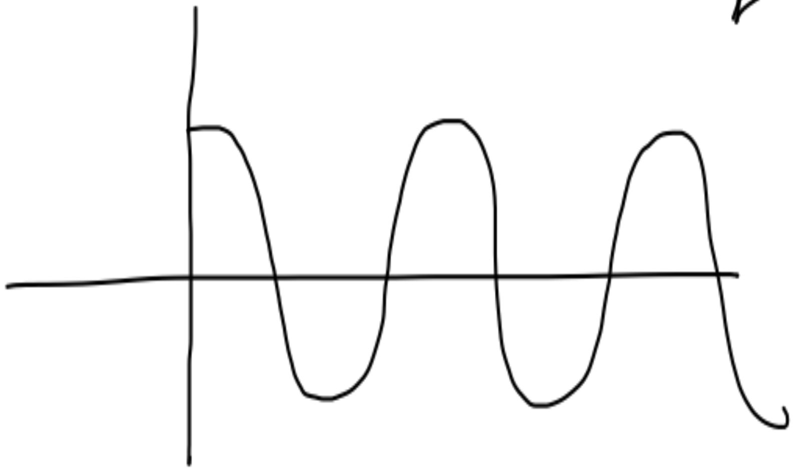
$$sF(s) = \frac{4s}{s^2+4}$$



$$\lim_{s \to 0} sF(s) = \frac{4s}{s^2+4} = 0$$

do not match

we can not apply final value theorem



6. Initial Value Theorem:

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$$

no stability cond'tn is required.

EX: $F(s) = \dfrac{5}{s(s^2+s+2)} \Rightarrow \lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s) = 0$

EX: $F(s) = \dfrac{4}{s^2+4} \Rightarrow \lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s) = 0$

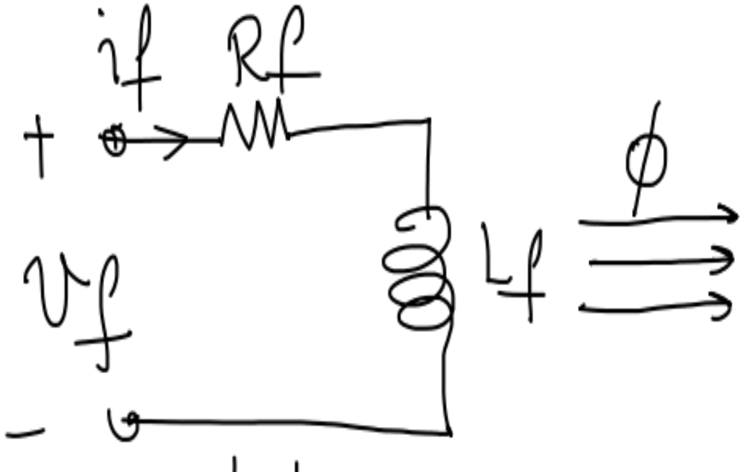
7. Convolution: $F_1(s) = \mathcal{L}\{f_1(t)\}$, $F_2(s) = \mathcal{L}\{f_2(t)\}$

$$\mathcal{L}\{f_1(t) * f_2(t)\} = \mathcal{L}\left\{\int_{-\infty}^{\infty} f_1(t-\tau) f_2(\tau)\, d\tau\right\}$$

$$= \mathcal{L}\left\{\int_{-\infty}^{+\infty} f_1(\tau) f_2(t-\tau)\, d\tau\right\} = F_1(s) \cdot F_2(s)$$

Ex: (Transfer function of DC Motor)

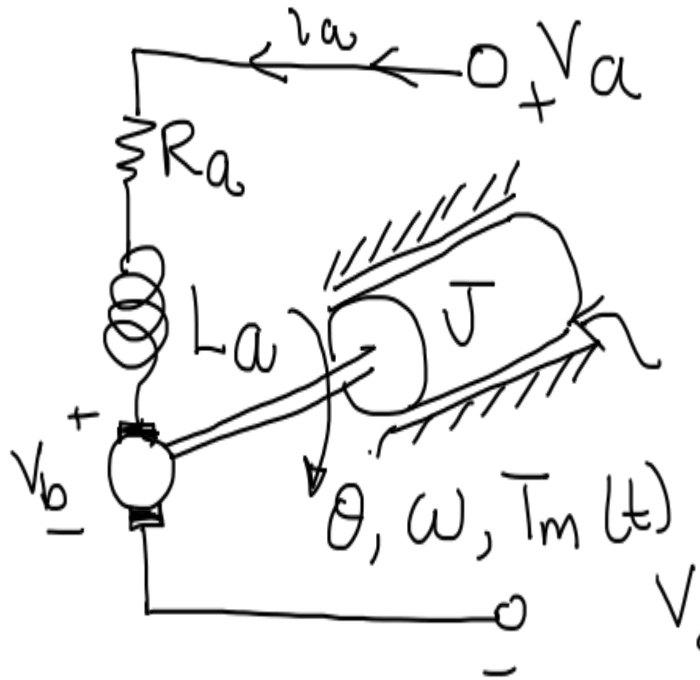
- A DC motor converts direct current (DC) electrical energy into rotational mechanical energy.
- Because of its features, such as high torque, speed controllability in a wide range, portability, well-behaved speed-torque characteristics, DC motors are widely used in numerous control applications such as robotic manipulators, transport systems, disk drives, machine tools etc.



stator part

Fieldpart and can replace this part by a permanent magnet.

$\Rightarrow$ PMDC motor



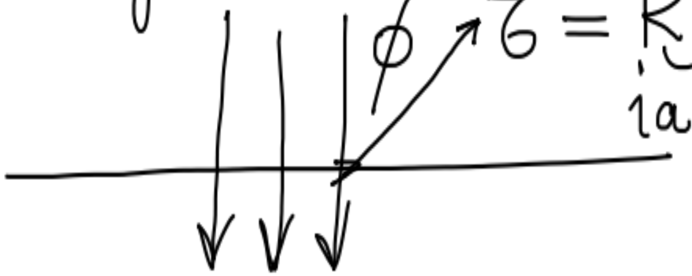
L_a: armature inductance of coil

R_a: armature resistance

$$V_b = K_b \cdot \omega$$

i_a: armature current

Recall: A coil placed in a magnetic field undergoes a torque



$$\tau = \underbrace{K \cdot \phi}_{K_i} \cdot i_a$$

moment (torque) constant

If i_f is constant, we have an armature controlled DC motor. The torque $T_m(t)$ is given by

$$T_m(t) = \underbrace{K_i K_f i_f}_{\text{const.}} \cdot i_a = \underbrace{K_m}_{\text{moment (torque) const.}} \cdot i_a(t)$$

$$L \frac{d}{dt} i(t) = v_L(t)$$

$$L s I = V$$

① $T_m(t) = K_m i_a(t)$

② $V_a(s) = (R_a + s L_a) I_a(s) + V_b(s)$

Back EMF Voltage generated by the rotation of the shaft $\approx \omega$

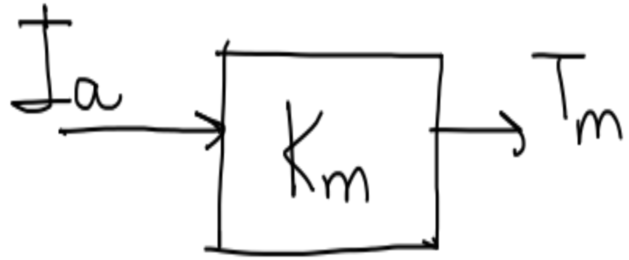
③ $V_b(s) = K_b \omega(s) = K_b s \theta(s)$

(4) $T_m(t) = (Js^2 + bs)\,\theta(s)$

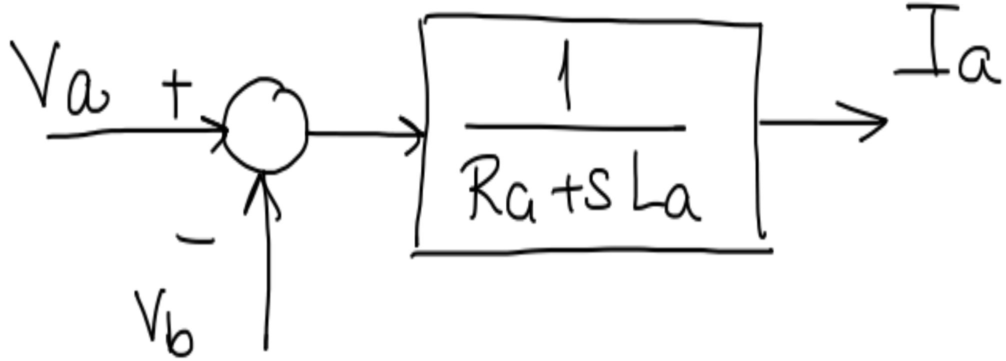
b: viscous friction const.

J: inertia of the motor

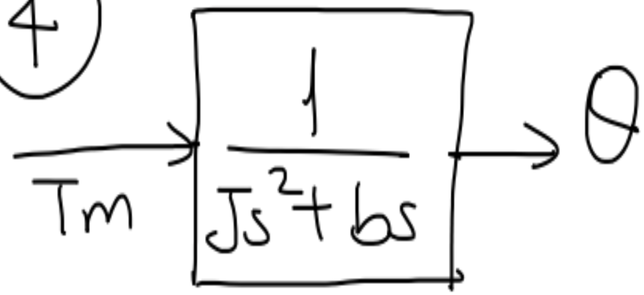
(1)



(2)

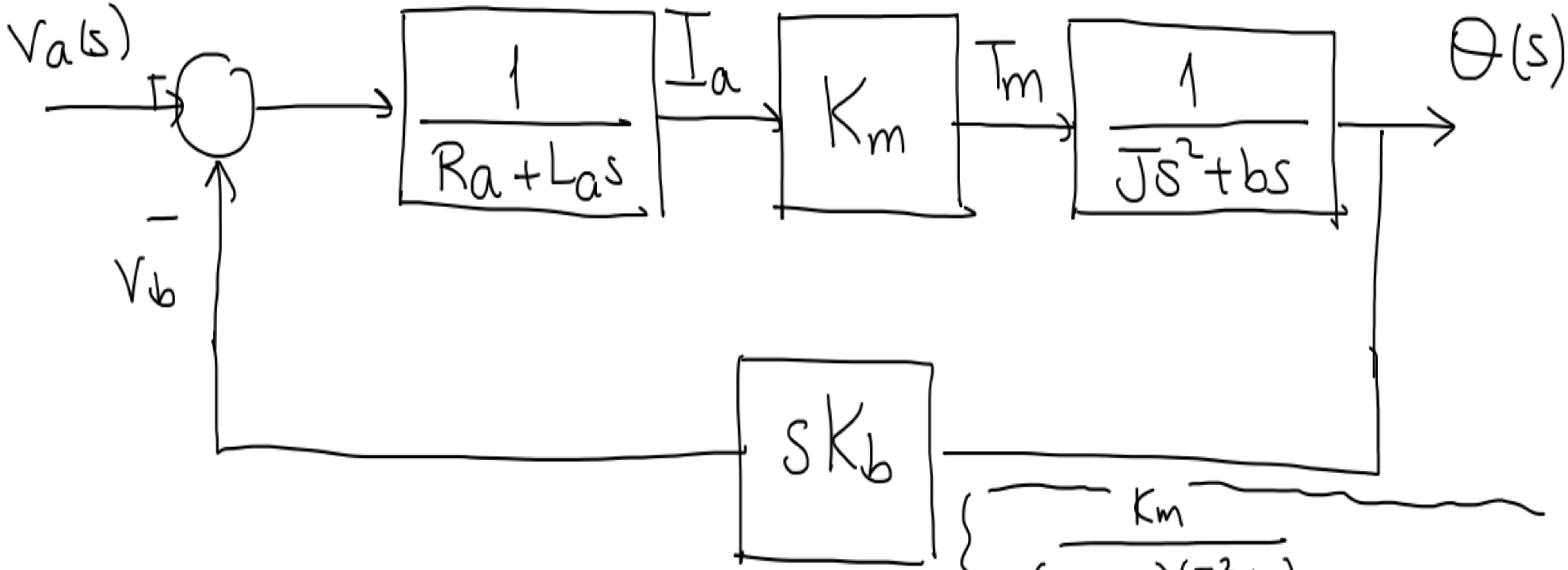


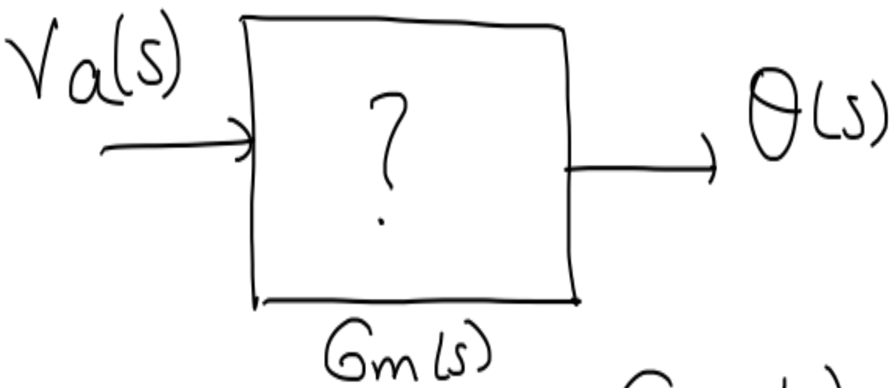
(4)



(3)

Block diagram of the system:

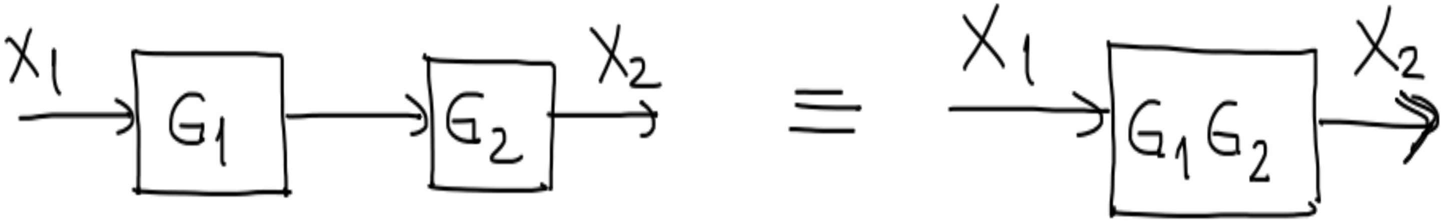


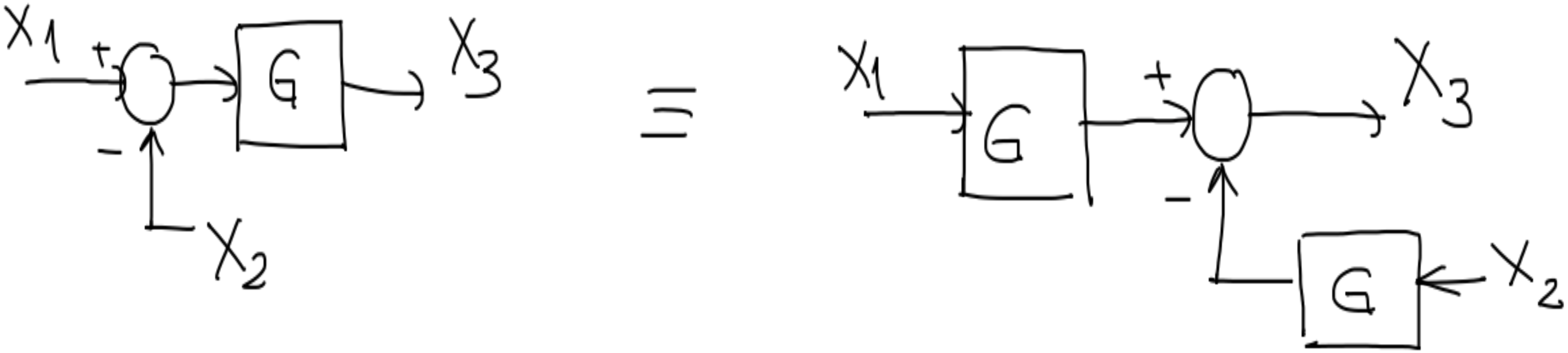


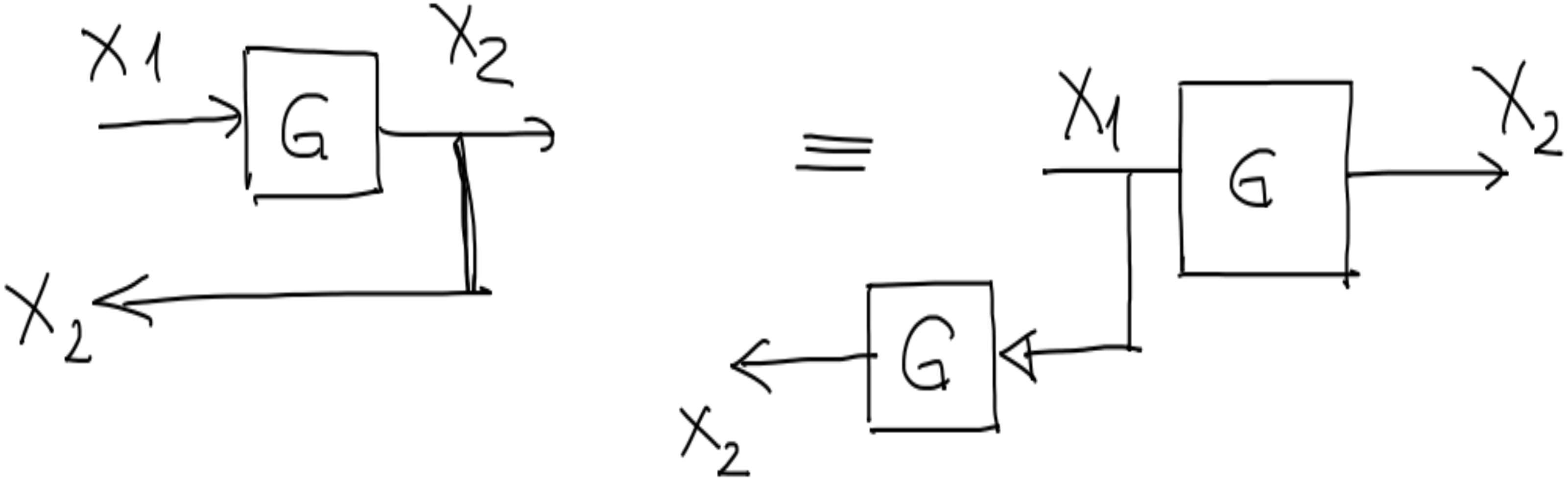
$$\frac{\frac{K_m}{(R_a + L_a s)(Js^2 + bs)}}{1 + \frac{K_m}{(R_a + L_a s)(Js^2 + bs)} s K_b}$$

$$G_m(s) = \frac{\Theta(s)}{V_a(s)} = \frac{K_m}{s\left[(R_a + L_a s)(J s + b) + K_b K_m\right]}$$

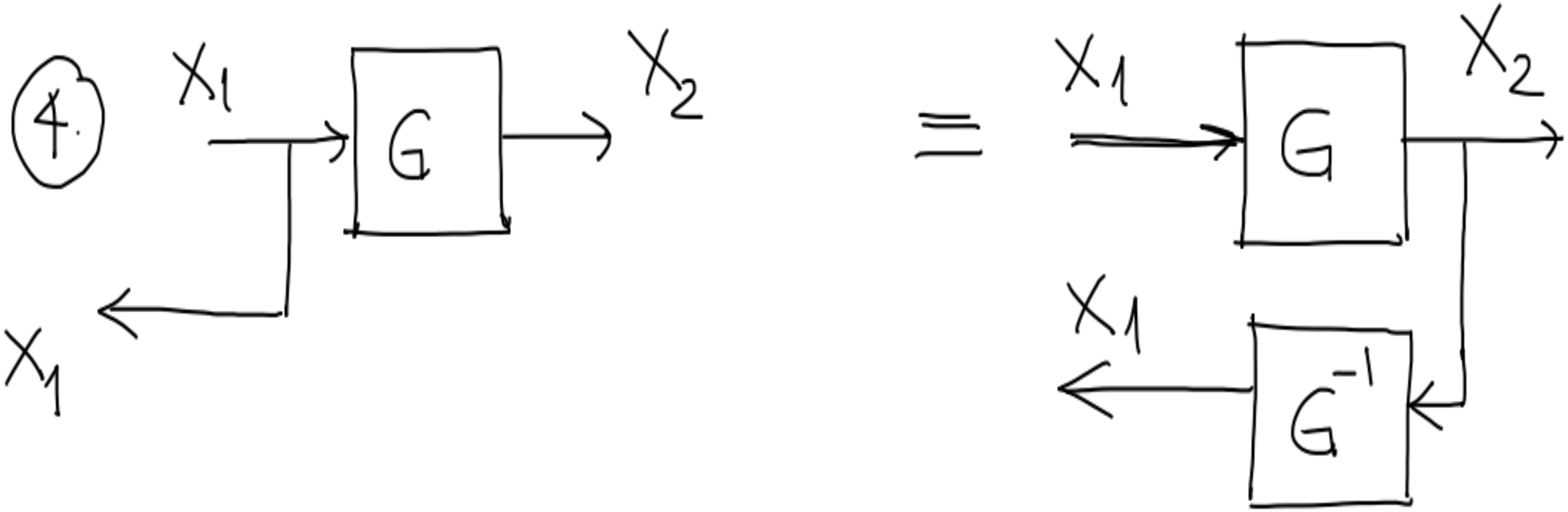
Block Diagram Reduction

1. 

2. 

3. 

4.



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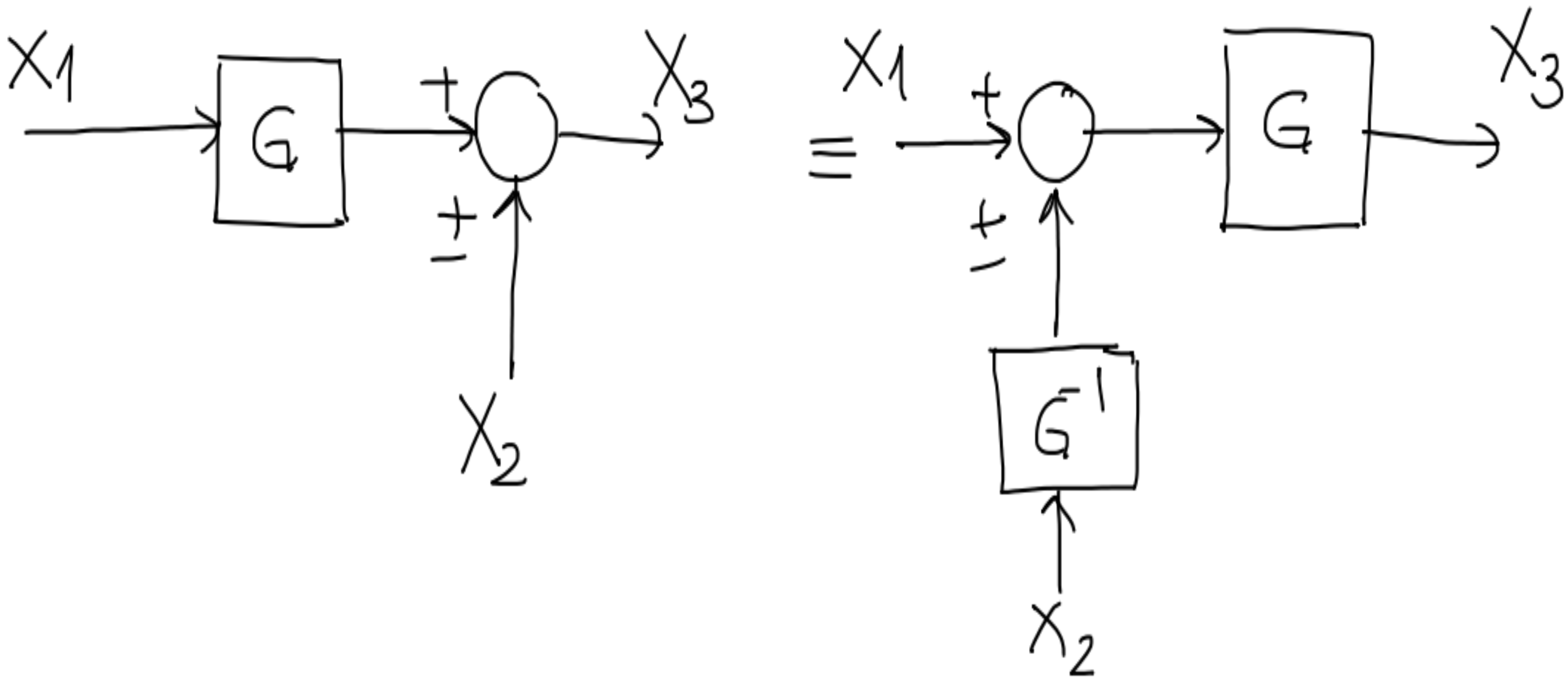

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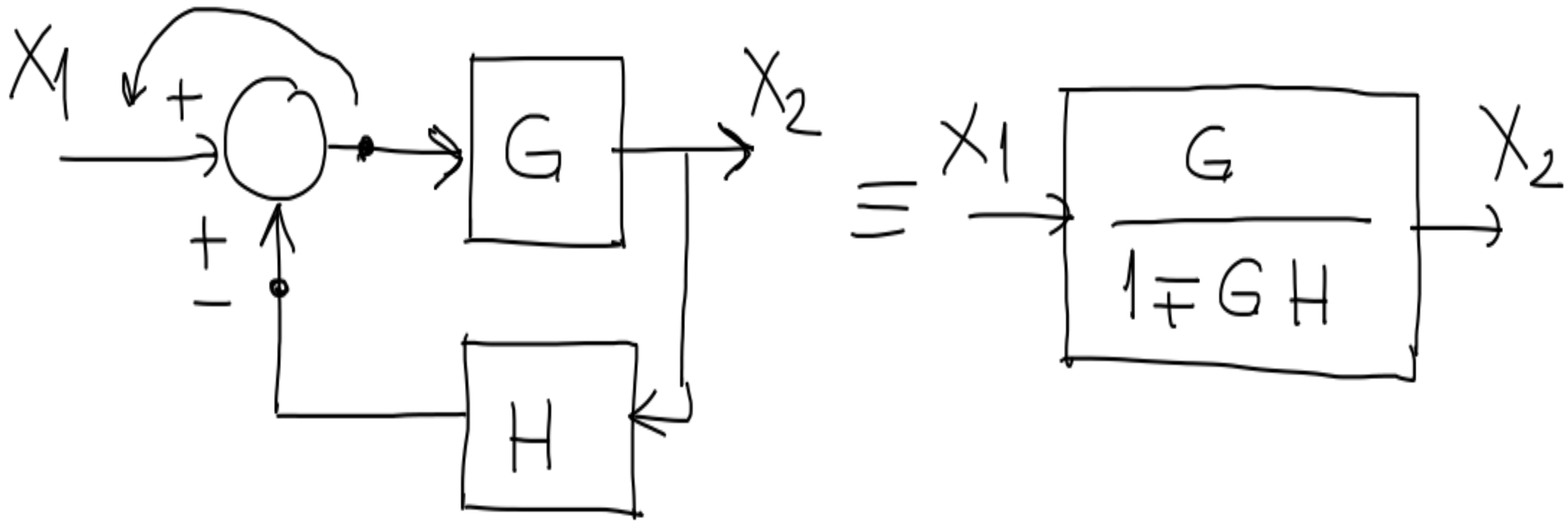
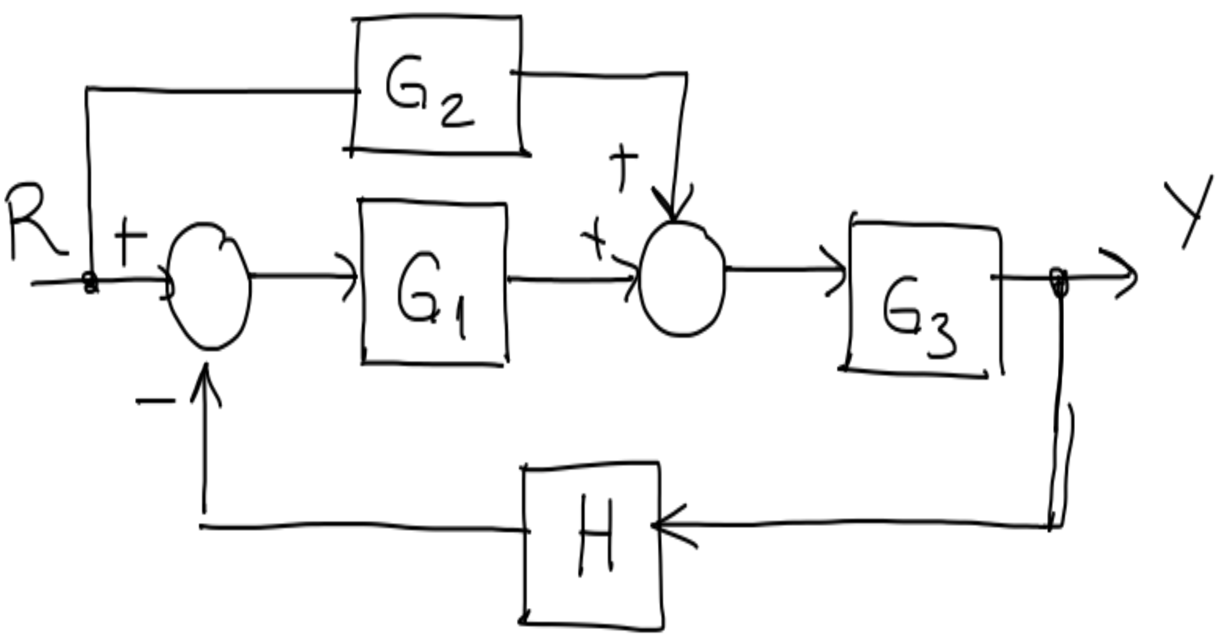


$$X_2 = G(1 \mp GH)^{-1} X_1$$



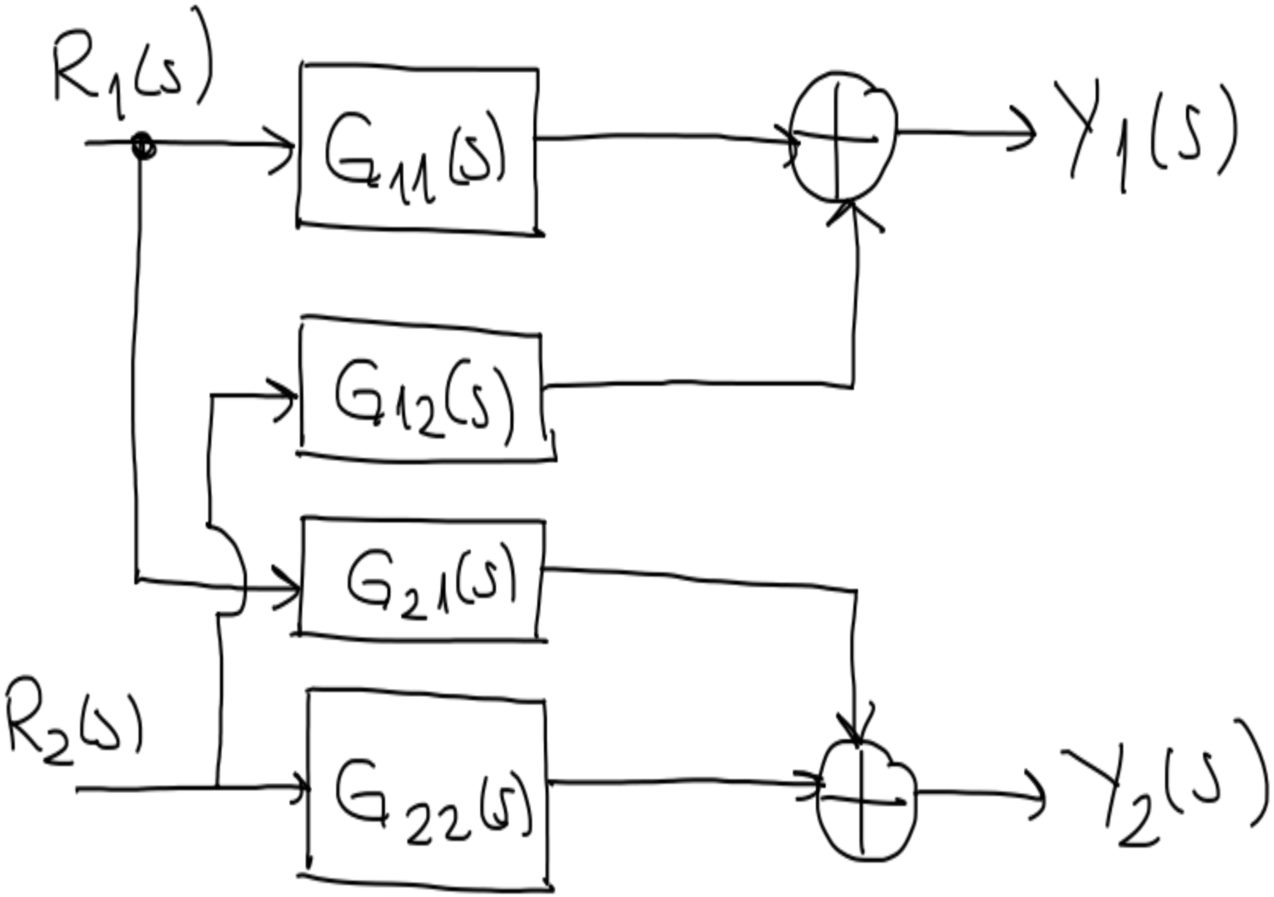
$$\frac{Y}{R} = ?$$

$$Y = G_3 G_1 (1 + G_1 G_3 H)^{-1} R + G_3 G_2 R$$

$$\frac{Y}{R} = \frac{G_3 G_1}{1 + G_1 G_3 H} + G_3 G_2$$

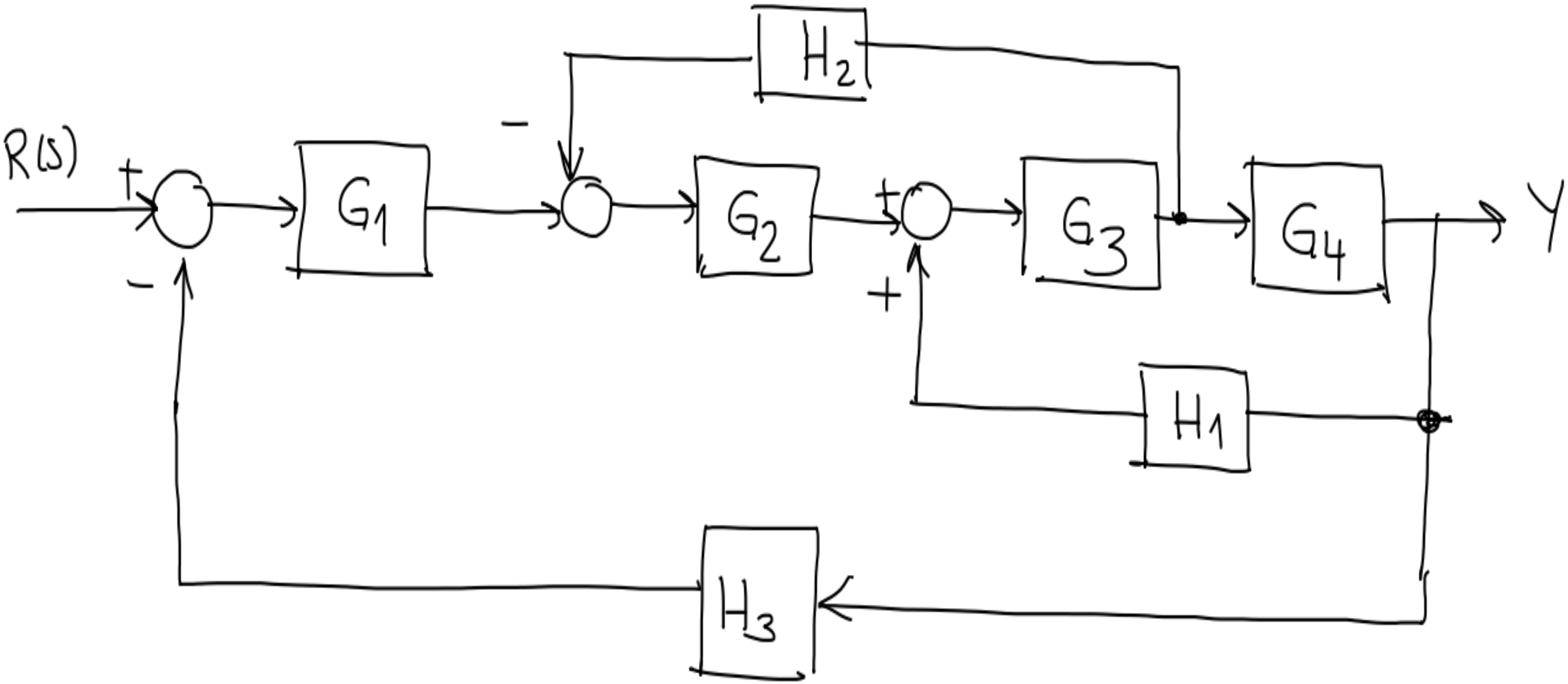
If our system has MI and MO ?

EX:

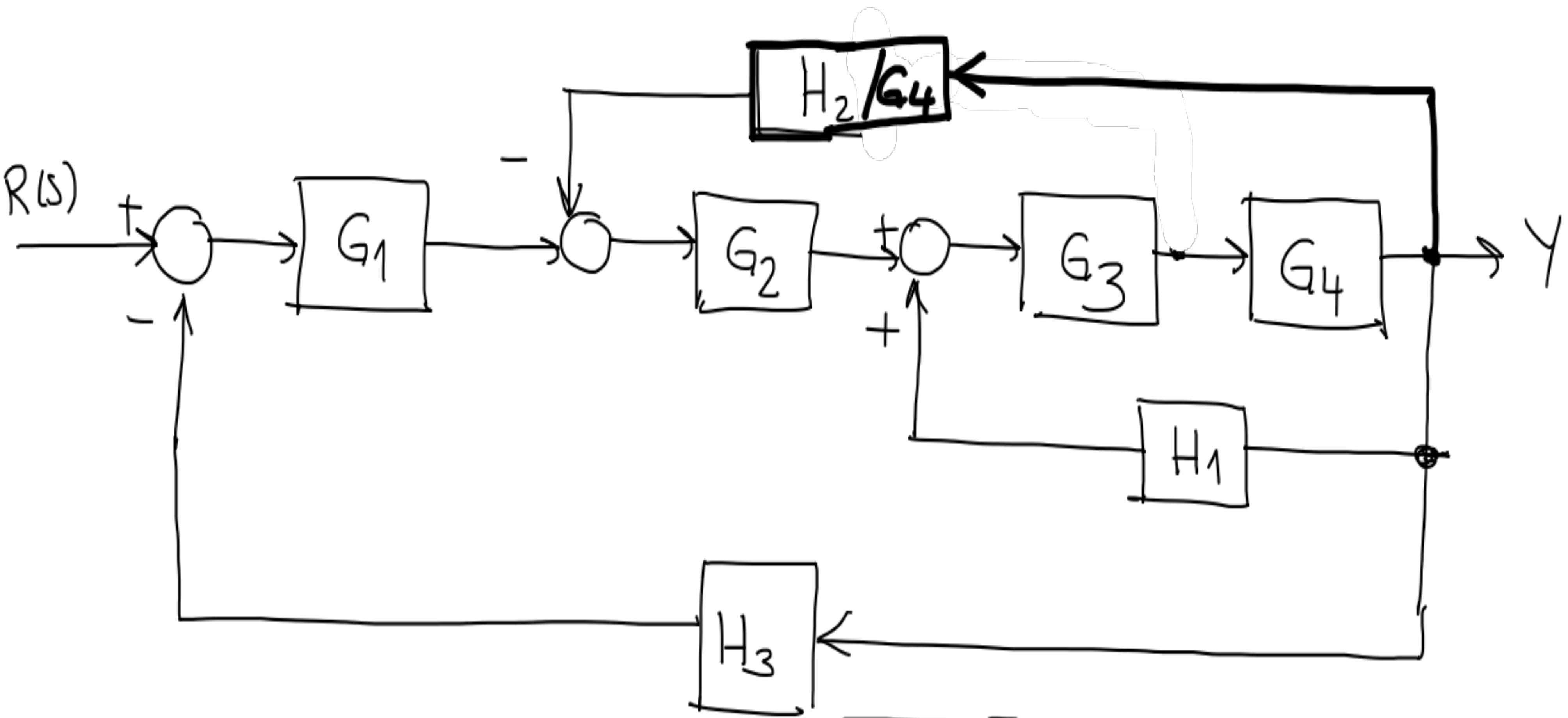


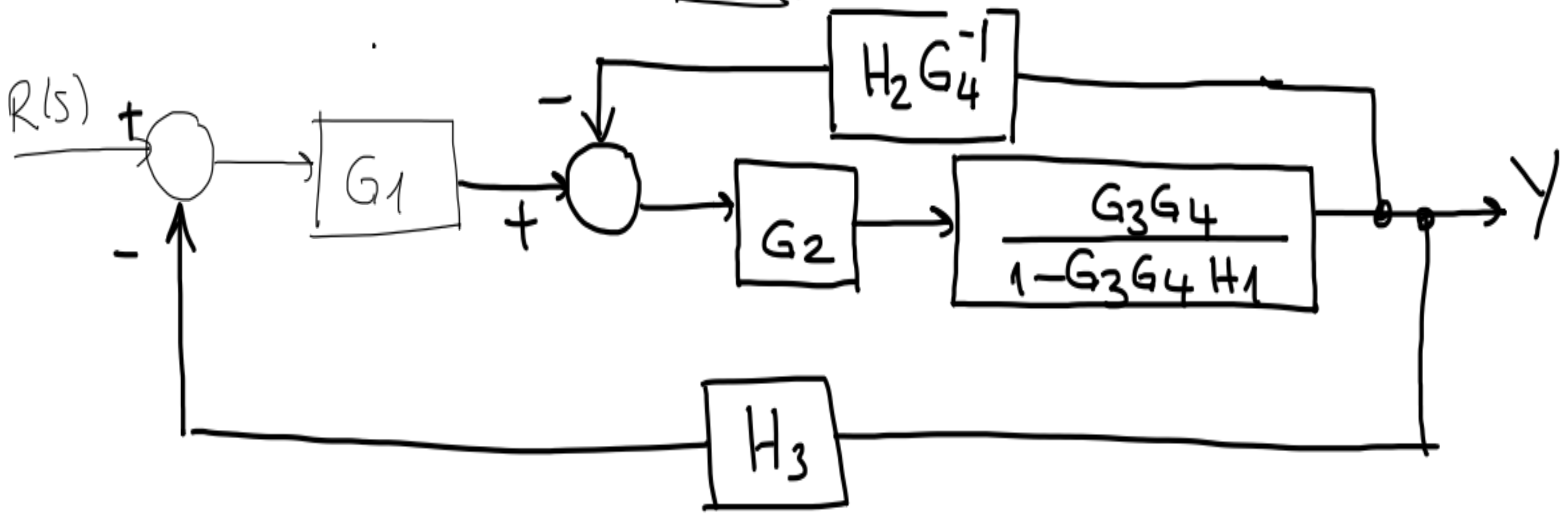
$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} R_1(s) \\ R_2(s) \end{bmatrix}$$

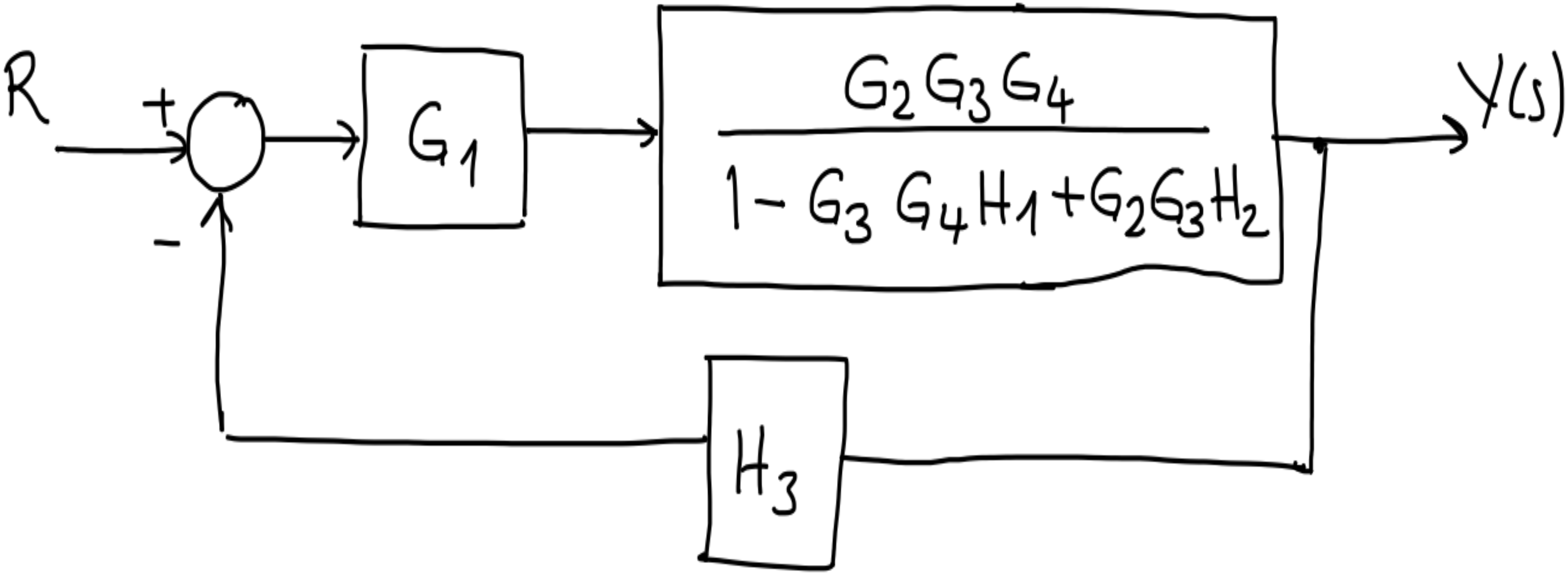
MIMO system

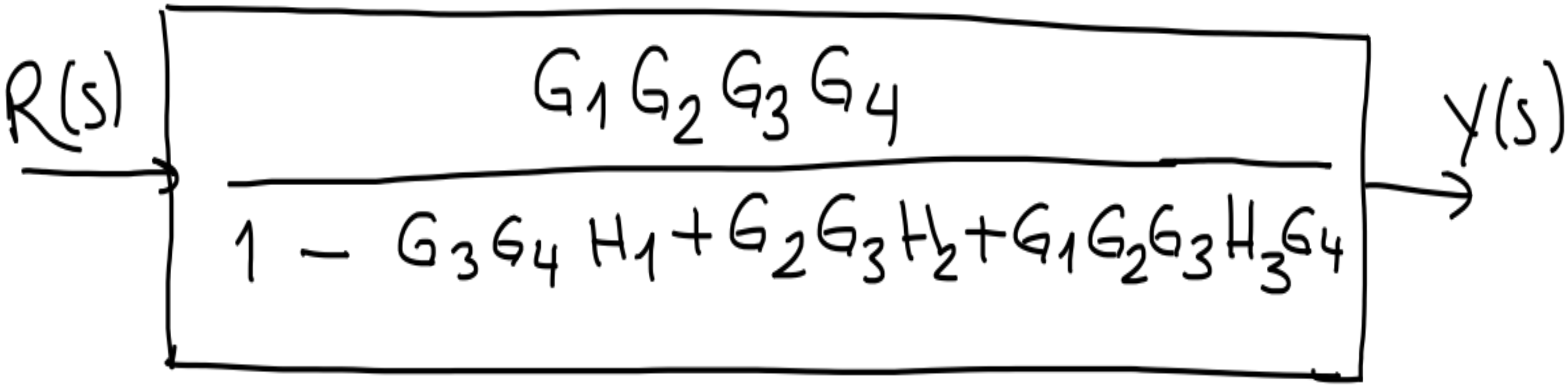


$$\frac{Y(s)}{R(s)} = ?$$ (use block diagram reduction)


R(s)
+
-
G1
G2
G3
G4
Y
H2/G4
H1
H3
+
+
-


R(s)
+
-
G1
+
-
G2
H2G4^-1
G3G4 / 1-G3G4H1
Y
H3


R
+
-
G_1
$\frac{G_2G_3G_4}{1-G_3G_4H_1+G_2G_3H_2}$
Y(s)
H_3


R(s)
$\frac{G_1G_2G_3G_4}{1-G_3G_4H_1+G_2G_3H_2+G_1G_2G_3H_3G_4}$
Y(s)

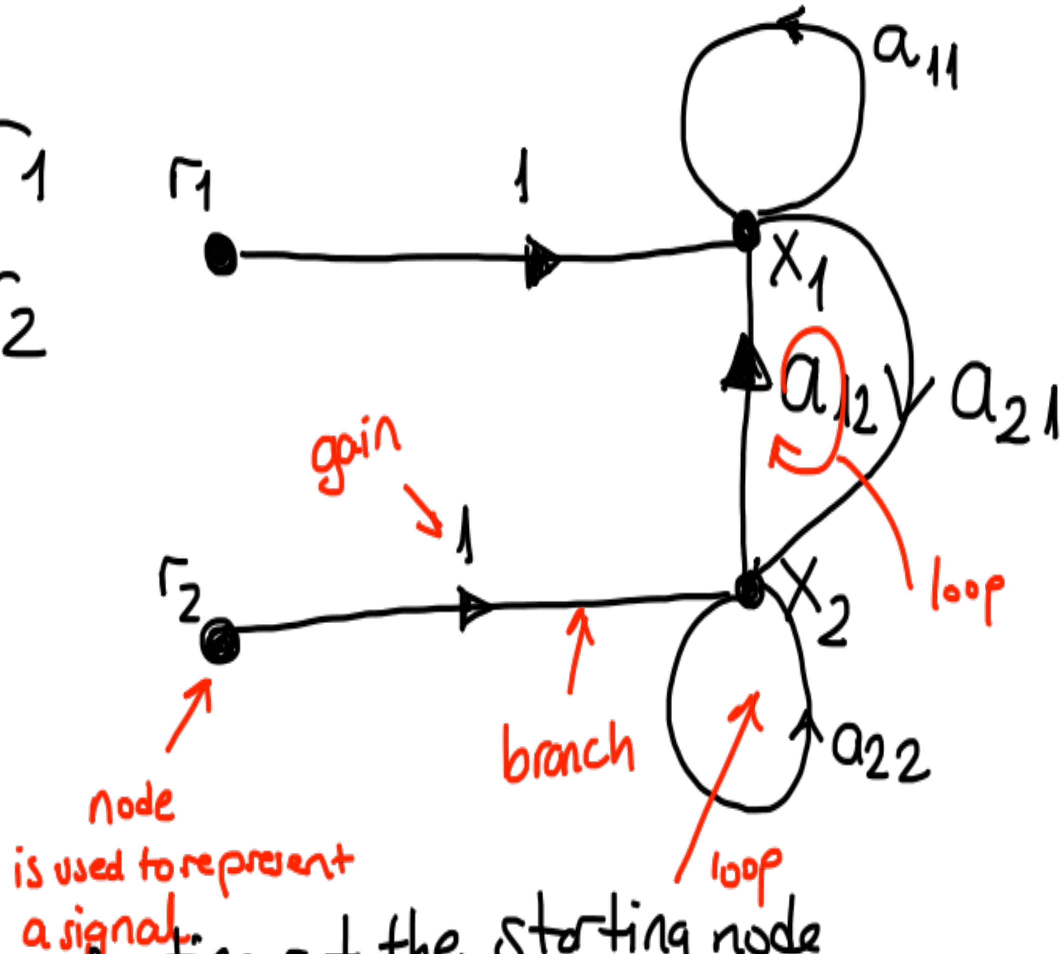
Signal Flow Graph Models

- For a system with reasonably complex interconnectionships, the block diagram reduction process is cumbersome.
- An alternative method for determining the relationship between system variables has been developed by Mason is based on graph theory
- A signal flow graph is a diagram consisting "nodes" that are connected by directed branches and it is a graphical representation of a system of linear relations.

Ex:

$$x_1 = a_{11} x_1 + a_{12} x_2 + r_1$$

$$x_2 = a_{21} x_1 + a_{22} x_2 + r_2$$



- A "Loop" is a closed path terminating at the starting node.
- Each signal or a variable is represented by a "NODE".
- Path between two nodes is called a "BRANCH"
- The value attached to each branch is called a "GAIN"
- A continuous sequence of branches that can be traversed from one signal to another is called a "PATH"

MASON'S GAIN FORMULA

Gain ↓ output signal ↓

$$M = \frac{Y_{out}}{Y_{in}} = \sum_{k=1}^{N} \frac{\delta_k \Delta_k}{\Delta}$$

↑ input signal

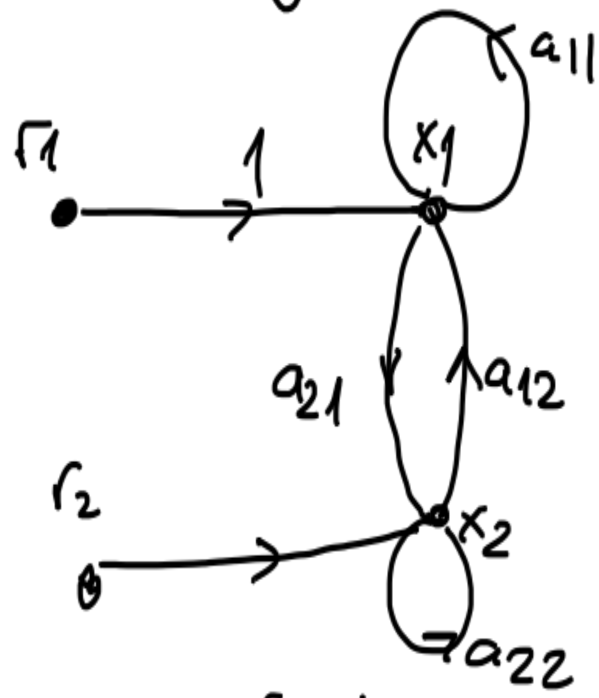
N = number of forward paths starting at Y_{in} and terminating at Y_{out}

δ_k : path gain of the k^{th} forward path originating at Y_{in} and terminating at Y_{out}

Δ : 1 − (sum of gains of all individual loops) + (sum of products of gains of all possible combos of <u>two</u> nontouching loops) − (sum of products of gains of all possible combos of 3 nontouching loops) + (" " " 4 " ") − …

Δ_k : The Δ corresponding to that part of the graph that is nontouching with the k^{th} forward path.

EX:
$$x_1 = a_{11}x_1 + a_{12}x_2 + r_1$$
$$x_2 = a_{21}x_1 + a_{22}x_2 + r_2$$



$$\frac{x_1}{r_1} = ?$$

$$\delta_1 = 1$$

$$M = \frac{x_1}{r_1} = \sum_{k=1}^{N} \frac{\delta_k \Delta_k}{\Delta} \longrightarrow N = 1 \Rightarrow M = \frac{x_1}{r_1} = \frac{\delta_1 \Delta_1}{\Delta}$$

$$\Delta = 1 - (a_{11} + a_{21} \cdot a_{12} + a_{22}) + (a_{11} \cdot a_{22})$$
$$\Delta_1 = 1 - (\quad a_{22} \quad)$$

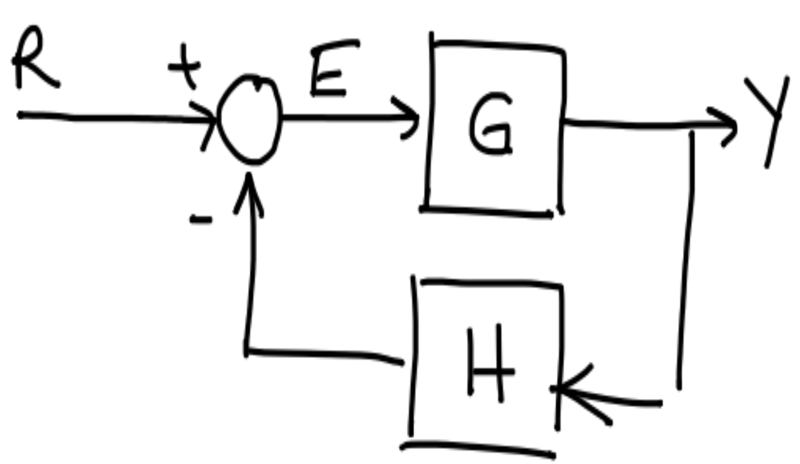
$$M = \frac{x_1}{r_1} = \frac{\Delta_1 \delta_1}{\Delta} = \frac{(1-a_{22})1}{1-(a_{11}+a_{12}a_{21}+a_{22})+a_{11}a_{12}}$$

HW=? show that $\frac{x_1}{r_2} = \frac{a_{12}}{\Delta}$, $\frac{x_2}{r_1} = \frac{a_{21}}{\Delta}$

$$\frac{x_2}{r_2} = \frac{1-a_{11}}{\Delta}$$

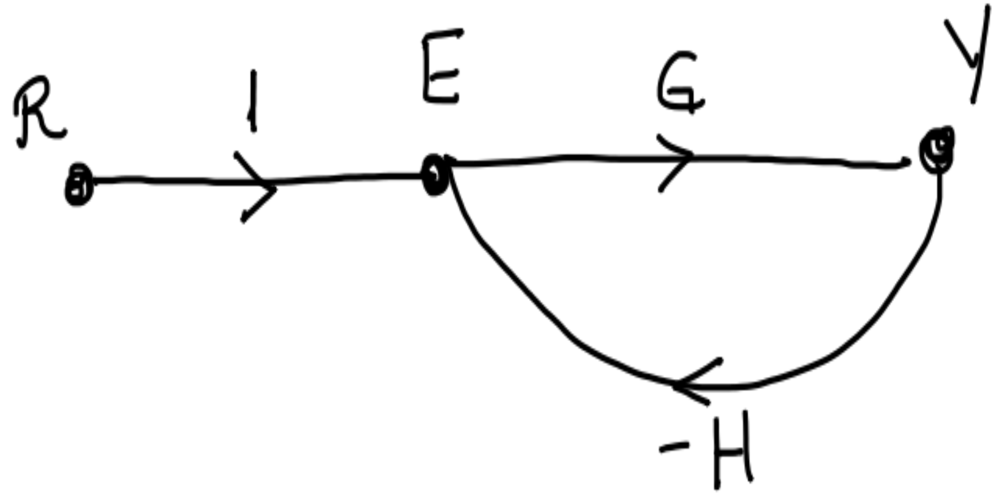
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} 1-a_{22} & a_{12} \\ a_{21} & 1-a_{11} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

EX:



$$\frac{Y}{R} = ?$$

$$\frac{Y}{R} = \frac{G}{1+GH}$$



$$\frac{Y}{R} = \sum_{k=1}^{N} \frac{\delta_k \Delta_k}{\Delta}$$

$$N = 1$$

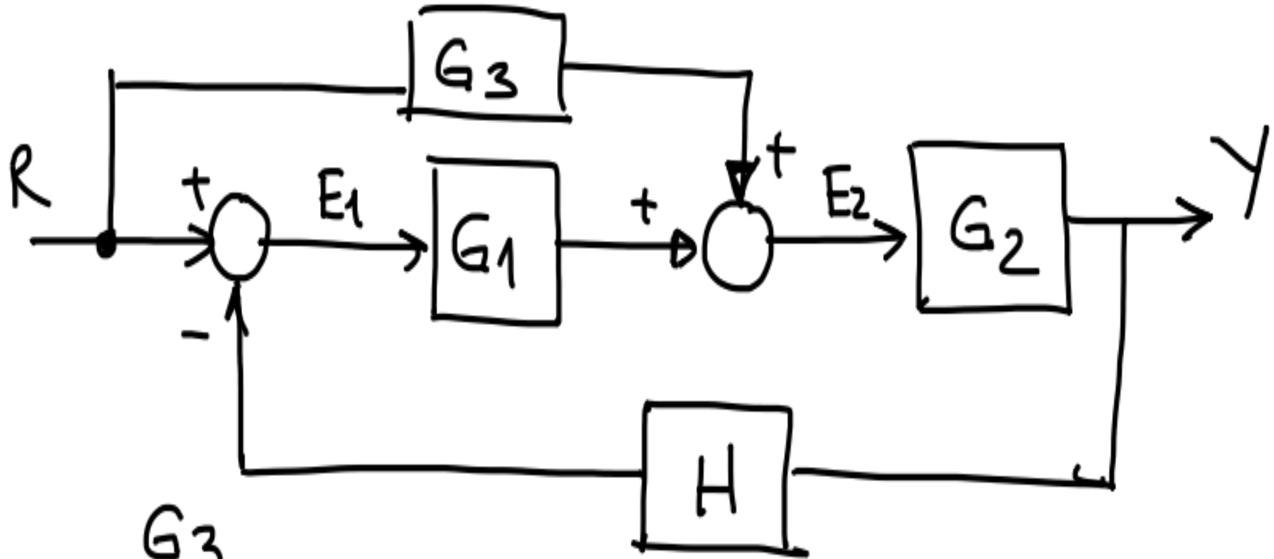
$$\delta_1 = G$$

$$\Delta = 1 - \left(-GH \right)$$

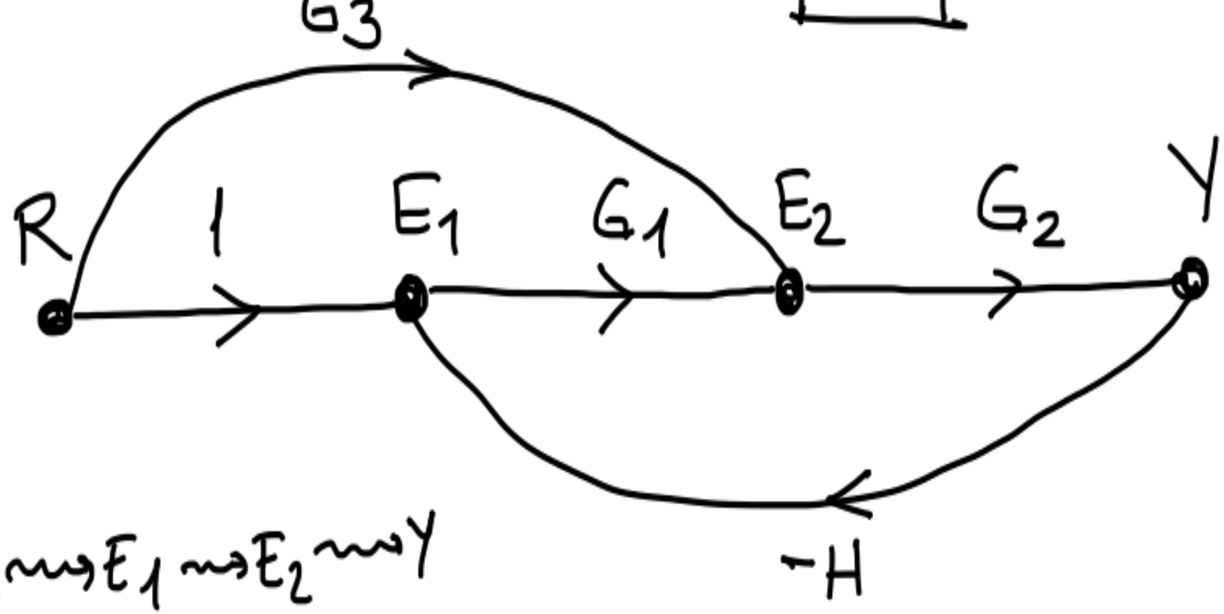
$$\Delta_1 = 1$$

$$\frac{Y}{R} = \frac{\delta_1 \Delta_1}{\Delta} = \frac{G \cdot 1}{1+GH} = \frac{G}{1+GH}$$

Ex:



$$\frac{Y}{R} = ?$$



$$\frac{Y}{R} = \sum_{k=1}^{N} \frac{\delta_k \Delta_k}{\Delta}$$

$N = 2$

$$\frac{Y}{R} = \frac{\delta_1 \Delta_1 + \delta_2 \Delta_2}{\Delta}$$

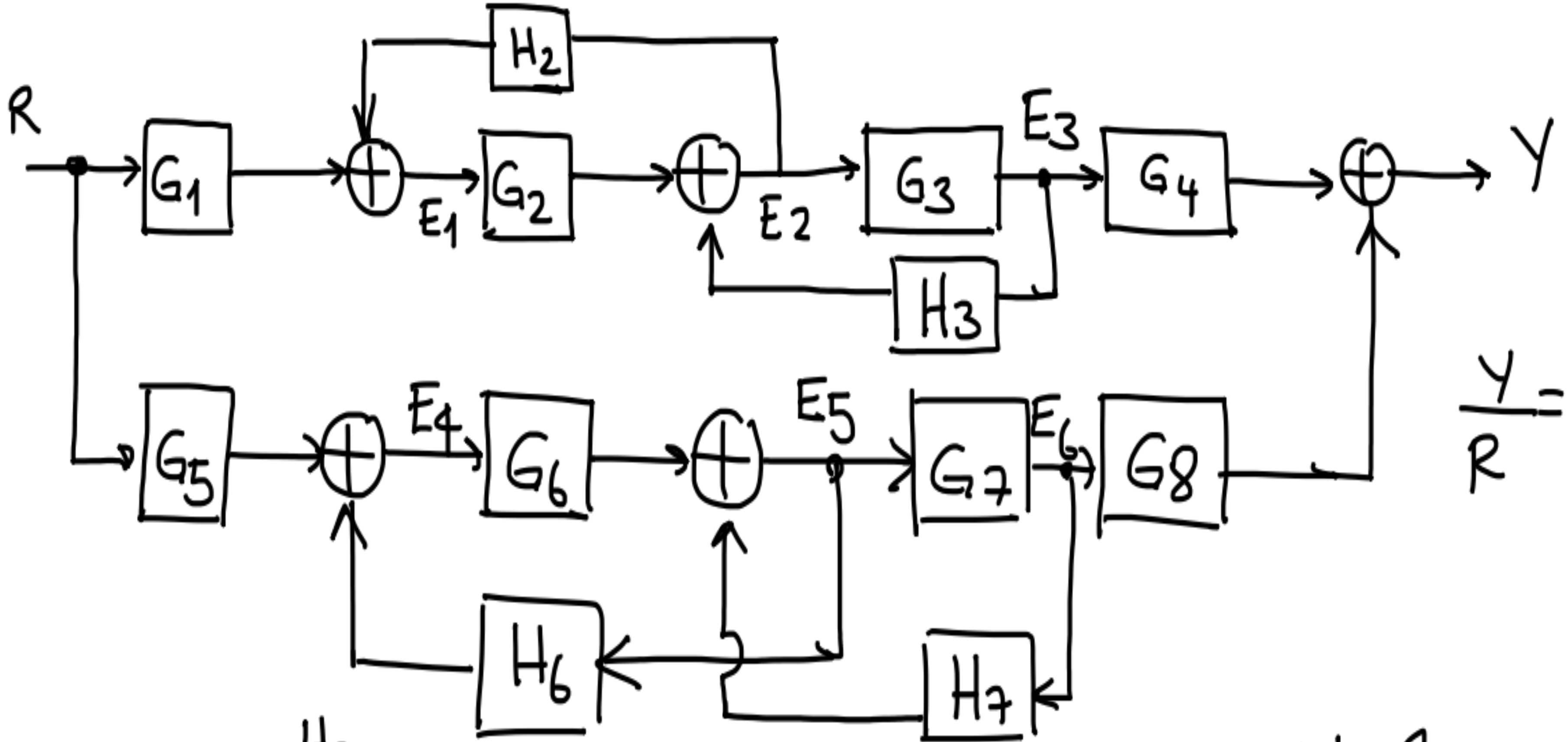
(1) $R \rightsquigarrow E_1 \rightsquigarrow E_2 \rightsquigarrow Y$

(2) $R \rightsquigarrow E_2 \rightsquigarrow Y$

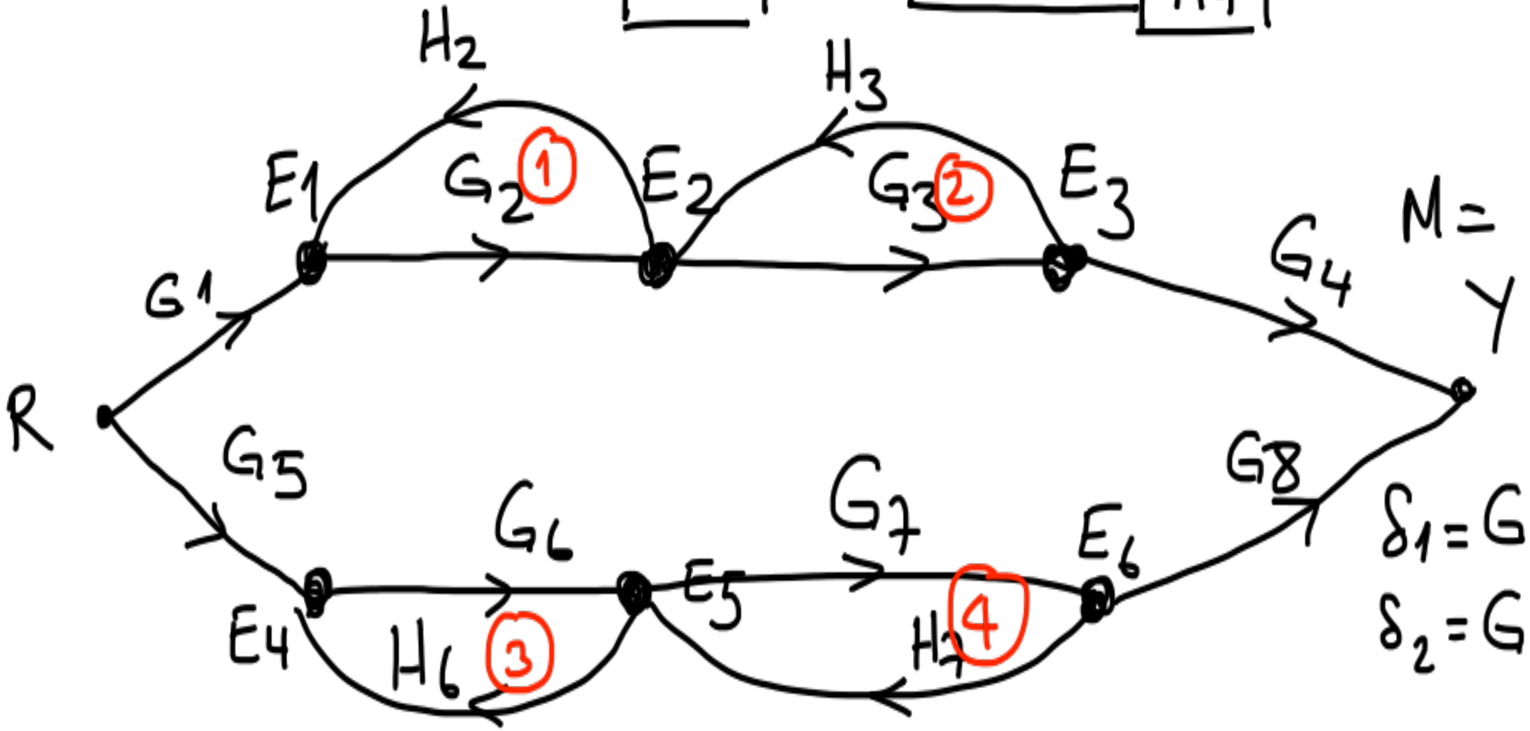
$\delta_1 = G_1 G_2$, $\delta_2 = G_3 G_2$ $\quad \Delta = \underline{1} - \left(- G_1 G_2 H\right)$, $\Delta_1 = 1$, $\Delta_2 = 1$

$$\frac{Y}{R} = \frac{G_1 G_2 + G_3 G_2}{1 + G_1 G_2 H}$$

EX:



$$\frac{Y}{R} = ?$$



$N = 2$

$$M = \frac{\Delta_1 \delta_1 + \Delta_2 \delta_2}{\Delta}$$

$\delta_1 = G_1 G_2 G_3 G_4$

$\delta_2 = G_5 G_6 G_7 G_8$

$$\Delta = 1 - (G_2 H_2 + G_3 H_3 + G_6 H_6 + G_7 H_7)$$
$$+ (G_2 H_2 G_6 H_6 + G_2 H_2 G_7 H_7 + G_3 H_3 G_6 H_6 + G_3 H_3 G_7 H_7)$$

$$\Delta_1 = 1 - (G_6 H_6 + G_7 H_7)$$

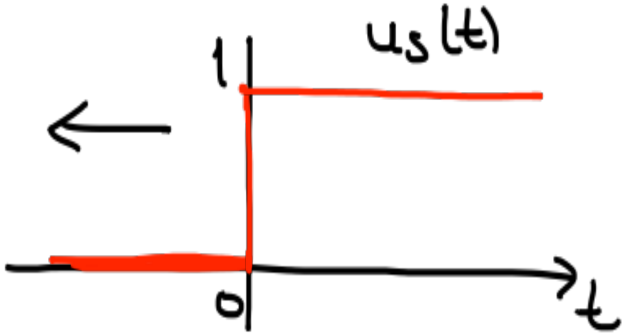
$$\Delta_2 = 1 - (G_2 H_2 + G_3 H_3)$$

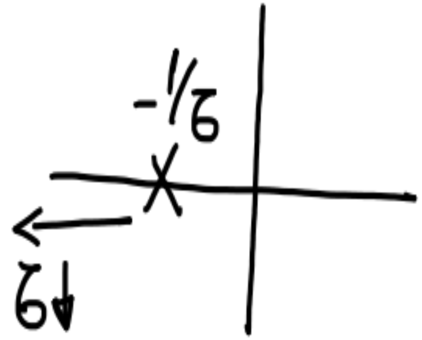
$$M = \frac{Y}{R} = \frac{\left[1 - (G_6 H_6 + G_7 H_7)\right] G_1 G_2 G_3 G_4 + (1 - G_2 H_2 - G_3 H_3) G_5 G_6 G_7 G_8}{\Delta}$$

First Order Dynamic Systems

Consider $\dfrac{Y(s)}{R(s)} = \dfrac{1}{\tau s + 1} = G(s)$ $\qquad R(s) \longrightarrow \boxed{\dfrac{1}{\tau s+1}} \longrightarrow Y(s)$

Assume $R(s) = \dfrac{1}{s}$ $\longleftarrow$





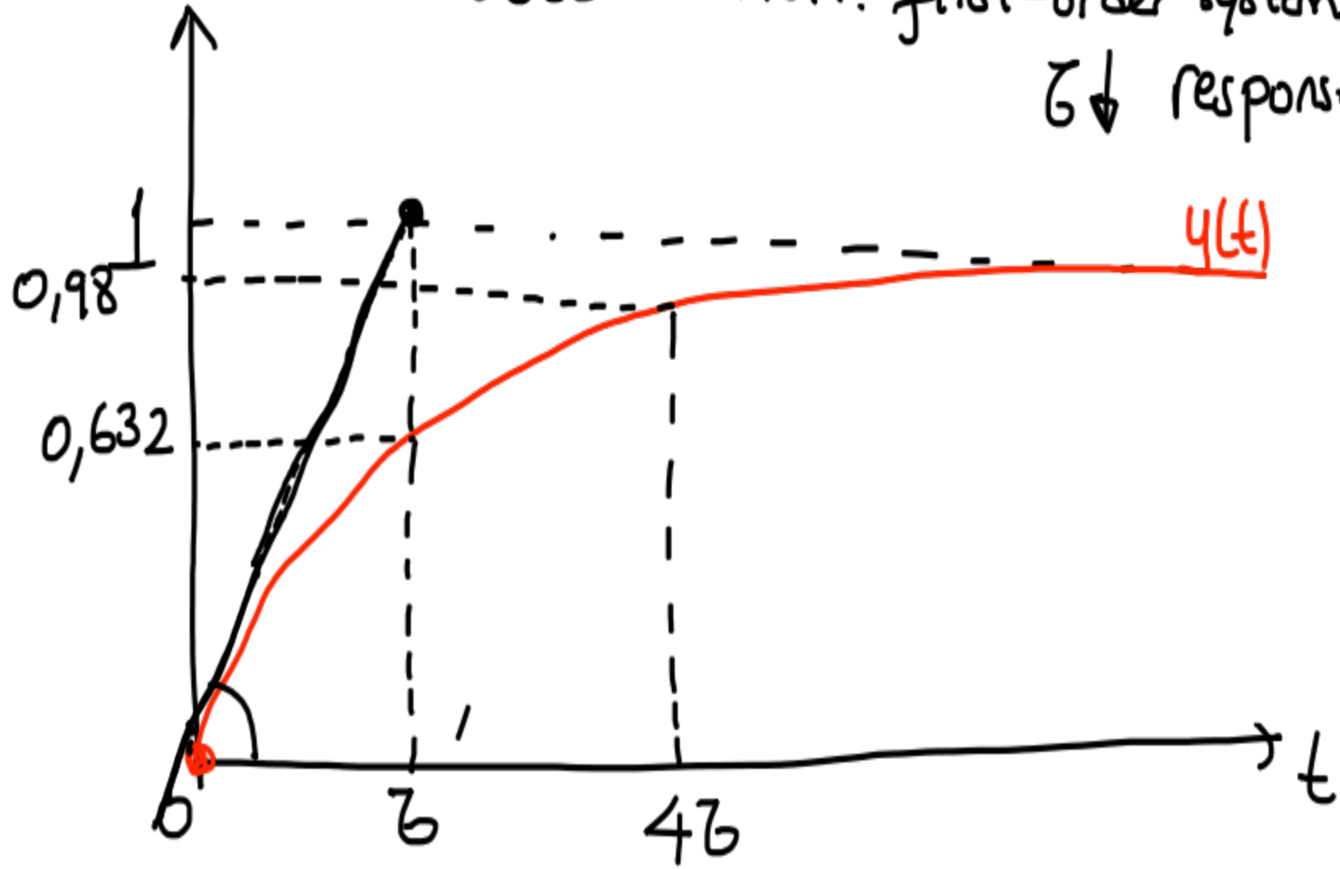
$$Y(s) = G(s)R(s) = \frac{1}{\tau s+1}\cdot\frac{1}{s} = \frac{A\,{}^{=1}}{s} + \frac{B\,{}^{=-\tau}}{\tau s+1}$$

$$= \frac{1}{s} - \frac{1}{s + 1/\tau}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \left(1 - e^{-\frac{1}{\tau}t}\right)u_s(t)$$

• observation: first-order systems can't make overshoots.

$\tau \downarrow$ response speed $\uparrow$



$y(0) = 0$

$y(\infty) = 1$

$y(t) = \left(1 - e^{-\frac{1}{\tau}t}\right) u_s(t)$

$t = \tau \rightarrow y(\tau) = ?$

$y(\tau) = \left(1 - e^{-1}\right) = 0,632$

$y(4\tau) = \left(1 - e^{-4}\right) = 0,98$

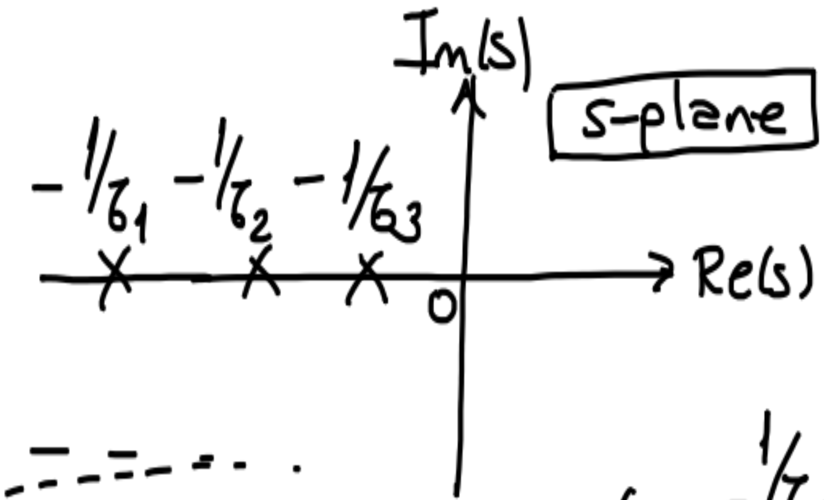
τ = time-constant

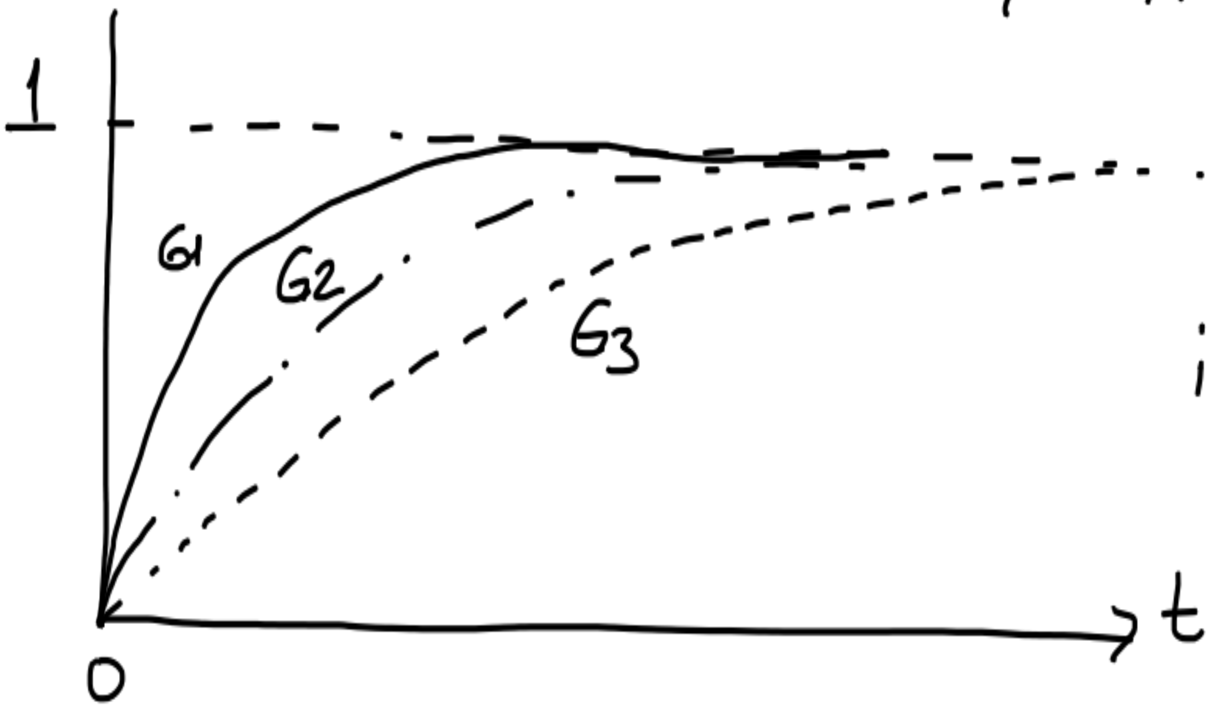
$t \geq 4\tau$ steady-state

$\frac{d}{dt} y(t) \Big|_{t=0} = \frac{1}{\tau} e^{-1/\tau t} \Big|_{t=0} = 1/\tau$

$$G_1 = \frac{1}{\tau_1 s + 1} \qquad G_2 = \frac{1}{\tau_2 s + 1} \qquad G_3 = \frac{1}{\tau_3 s + 1}$$

$$\tau_3 > \tau_2 > \tau_1$$



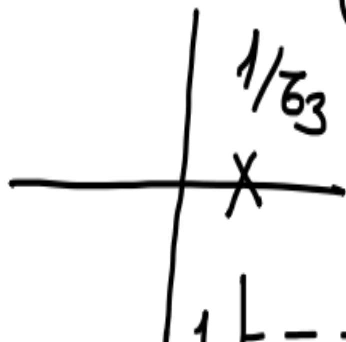


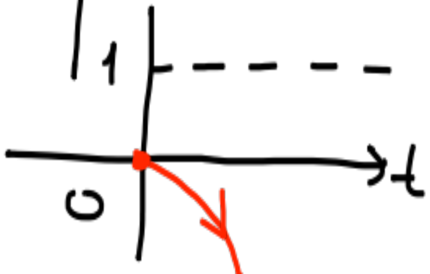
$$y(t) = \left(1 - e^{-1/\tau_3}\right) u(t)$$

if $\tau_3 < 0$

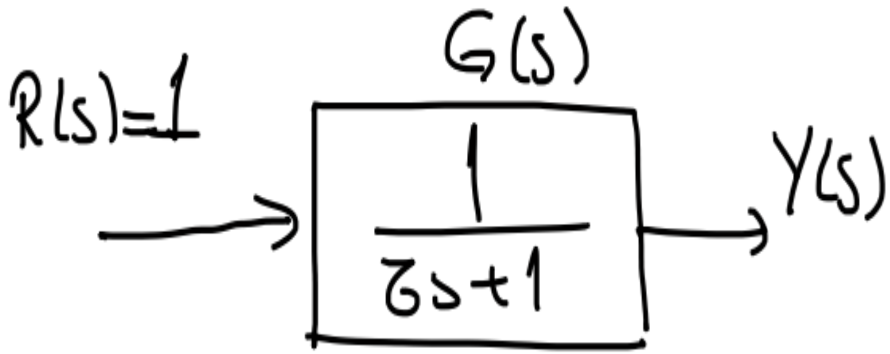
$$= \left(1 + e^{a}\right) u(t)$$

$a > 0$





Unit Impulse Response of a 1st order System:



$$Y(s) = G(s)\,R(s) \qquad R(s) = 1$$

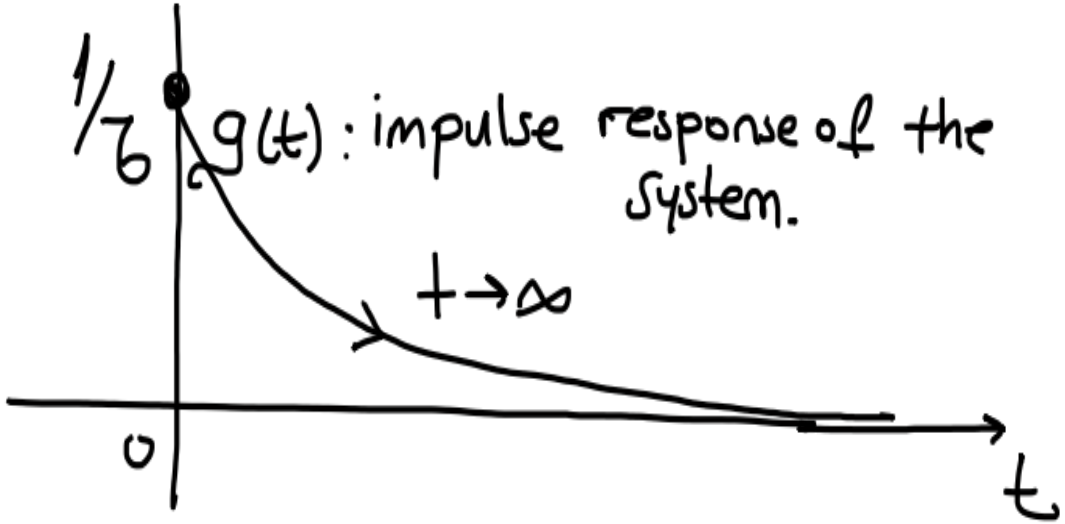
$$Y(s) = G(s)$$

$$Y(s) = G(s) = \frac{1}{\tau s + 1} = \frac{1/\tau}{s + 1/\tau}$$

$$y(t) = \frac{1}{\tau} e^{-1/\tau\, t} \cdot u_s(t)$$

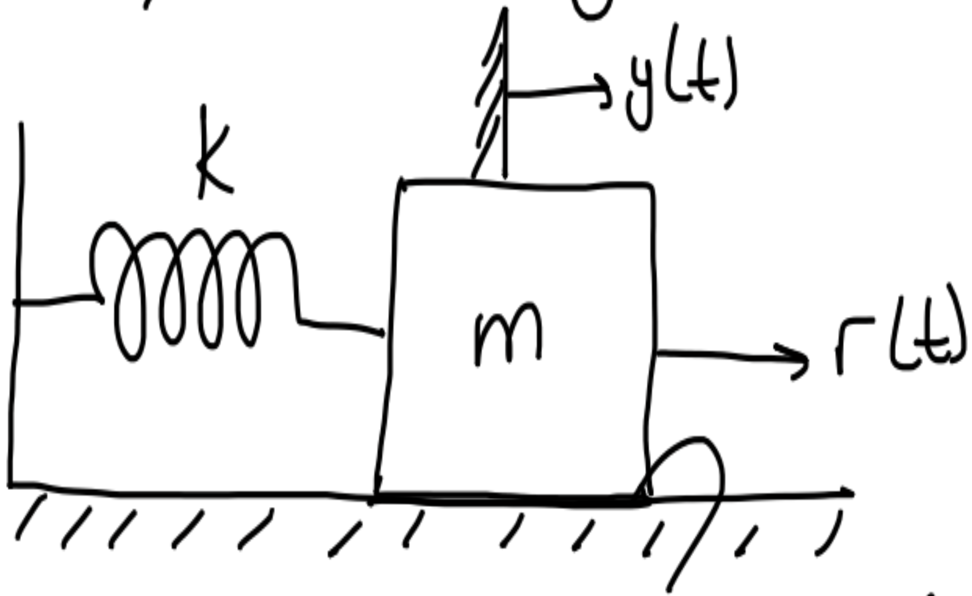
$$\downarrow$$

$$g(t)$$



2^{nd} Order Systems

Consider a spring-mass system with mass m friction b, and spring constant k



b: viscous friction cnst.

Eqn of motion:

$$m\ddot{y} = r - ky - b\dot{y}$$

Let's take the Laplace Transform of both sides by assuming all initial cond'ns are identical to zero.

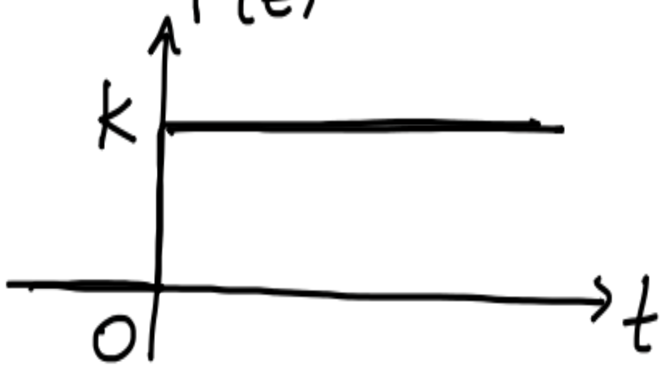
i.e., $y(0)=\dot{y}(0)=0$

$$m\ddot{y}=r-ky-b\dot{y} \xRightarrow{\mathcal{L}} ms^2\,Y(s)=R(s)-kY(s)-bsY(s)$$

$$\Rightarrow \quad Y(s)=\frac{1}{ms^2+bs+k}\cdot R(s)$$

$r(t) \longrightarrow R(s)=\frac{k}{s}$

Let's apply a constant input



$$Y(s) = \frac{1}{ms^2+bs+k} \cdot \frac{k}{s} = \frac{k/m}{\left(s^2 + b/m\, s + k/m\right)s}$$

$$y(\infty) = \lim_{s\to 0} sY(s) = 1$$

$$\xrightarrow{\mathcal{L}^{-1}} y(t) = ?$$

$$\frac{k}{m} =: \omega_n^2$$

$$\frac{b}{m} =: 2\xi\omega_n$$

ξ : damping ratio

ω_n : natural frequency.

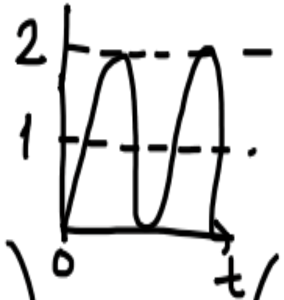
$$Y(s) = \frac{\omega_n^2}{s\left(s^2+2\xi\omega_n s+\omega_n^2\right)} \Rightarrow \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

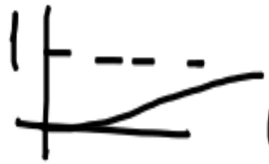
if $R(s) = 1/s$

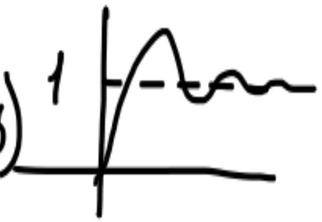
prototype second order system.

When ξ and ω_n varies, the response $y(t)$ changes ...

$$
y(t) = \begin{cases}
1 - \cos(\omega_n t) & (\xi = 0)\ (\text{undamped case}) \\[2ex]
1 - e^{-\omega_n t} - \omega_n t \cdot e^{-\omega_n t} & (\xi = 1)\ (\text{critically damped}) \\[2ex]
1 - \dfrac{\xi}{\alpha} e^{-\xi\omega_n t} \cosh(\omega_n \alpha t) & (\xi > 1)\ (\text{overdamped resp}) \\
\qquad \alpha = \sqrt{\xi^2 - 1} & \\[2ex]
1 - \dfrac{1}{\beta} e^{-\xi\omega_n t} \sin(\omega_n \beta t + \phi) & (0 < \xi < 1)\ (\text{underdamped case}) \\
\qquad \beta = \sqrt{1 - \xi^2} \qquad \phi = \cos^{-1}\xi &
\end{cases}
$$







For underdamped case:

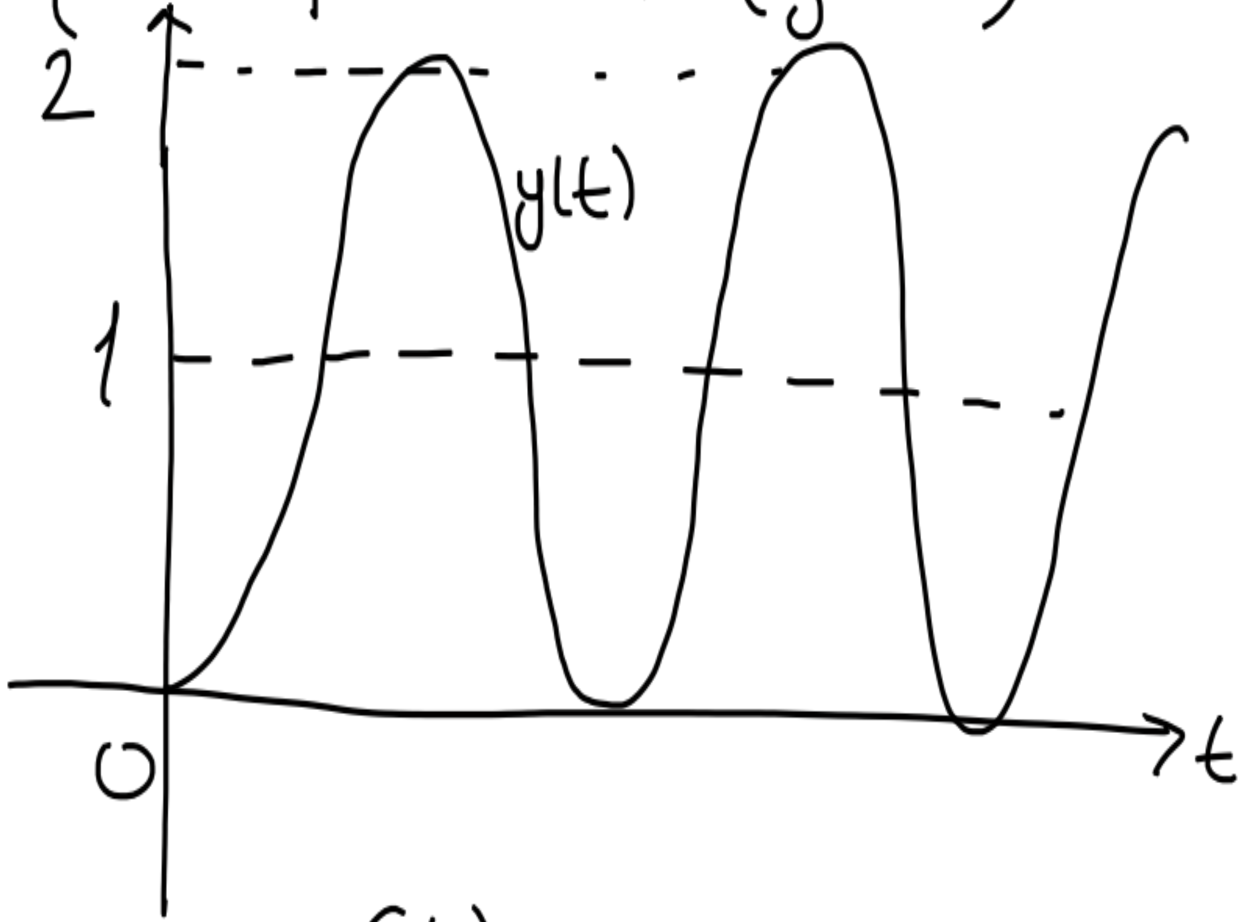
$$y(s) = \frac{1}{s} - \frac{s+2\xi\omega_n}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_n^2\sqrt{1-\xi^2}}$$

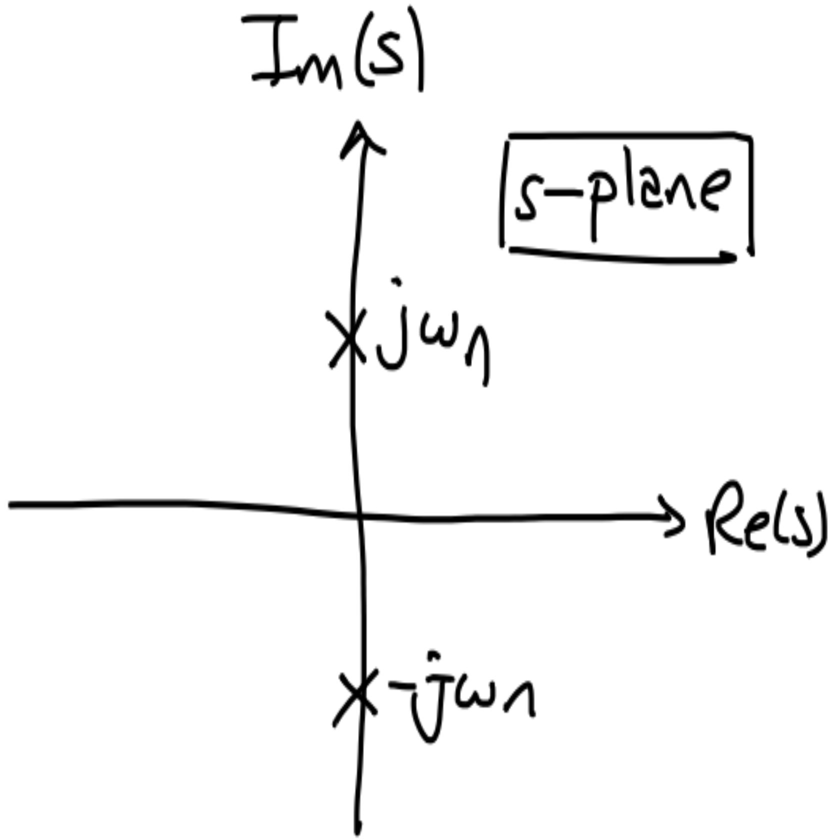
$$- \frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_n^2\sqrt{1-\xi^2}}$$

$$y(t) = 1 - e^{-\xi\omega_n t} \cdot \cos\omega_n\sqrt{1-\xi^2}\cdot t$$

$$- e^{-\xi\omega_n t} \cdot \frac{\xi}{\sqrt{1-\xi^2}} \cdot \sin\omega_n\sqrt{1-\xi^2}\, t$$

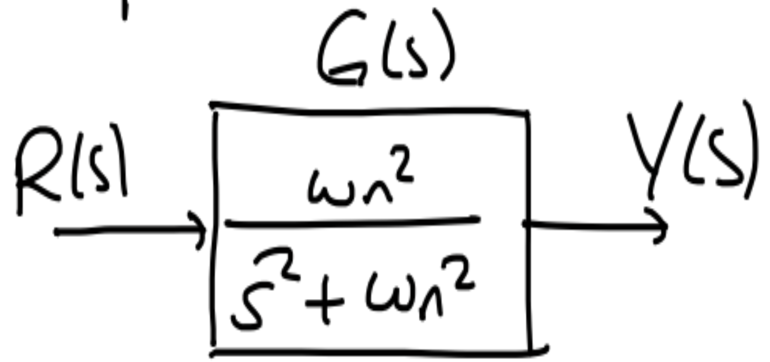
(Undamped case) ($\xi = 0$)



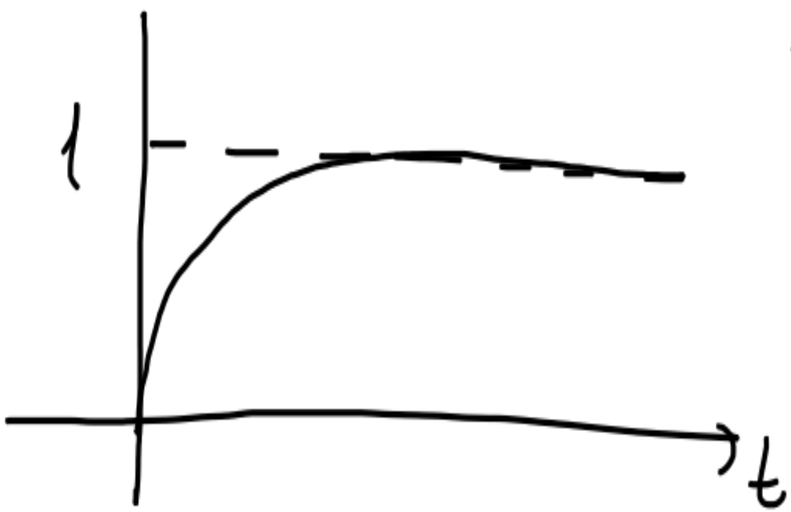


$$s^2 = -\omega_n^2$$

$$s_{1,2} = \mp j\omega_n$$

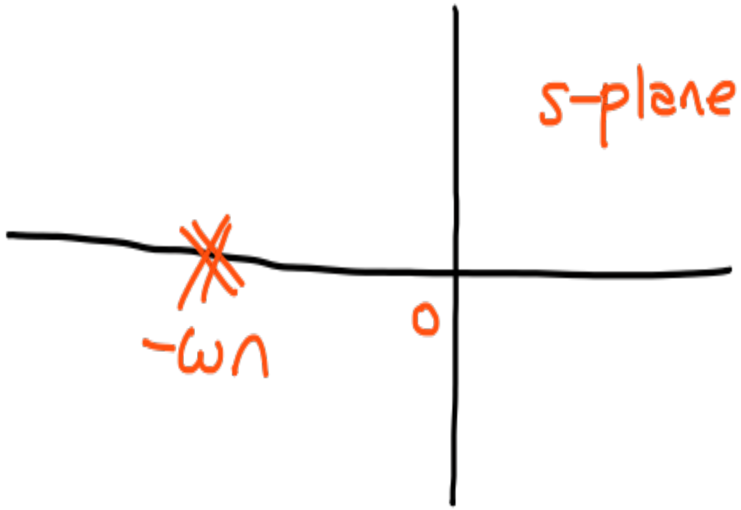


(Critically damped case, $\xi = 1$)



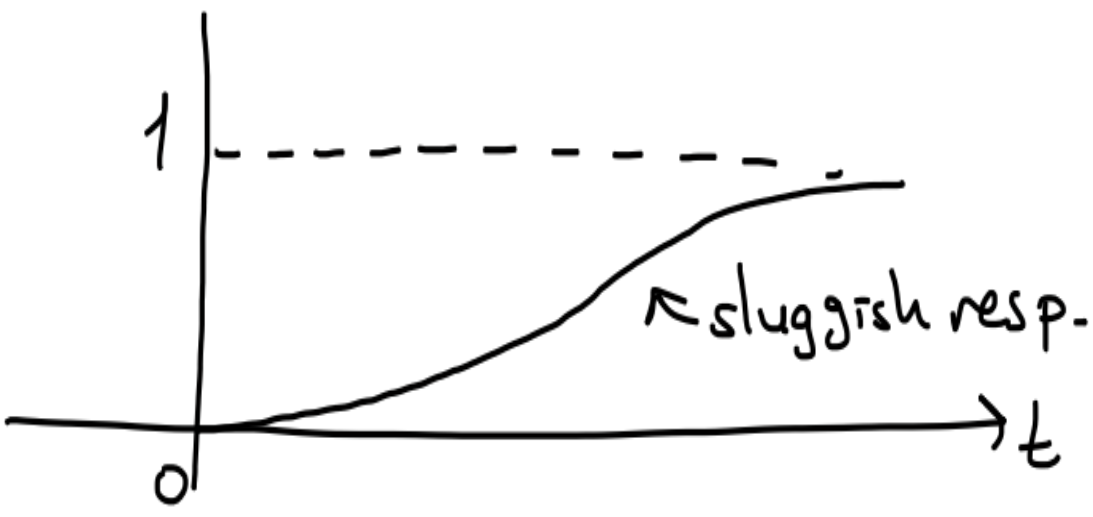
← Fastest resp. w/o overshoot.
(very similar to the resp. of a first order syst)

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$



$$G(s) = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} \overset{\xi=1}{=} \frac{\omega_n^2}{(s+\omega_n)^2}$$

(overdamped case) ($\xi > 1$)



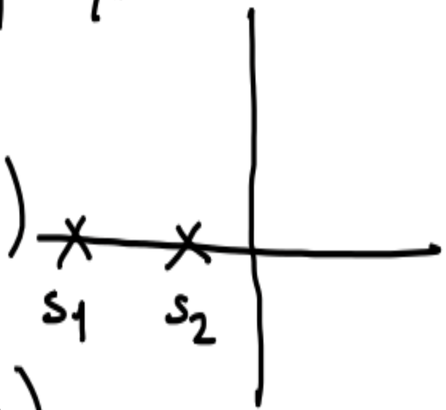
$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\Delta(s) = s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

characteristic poly.

$$s_{1,2} = -\xi\omega_n \mp \sqrt{\xi^2\omega_n^2 - \omega_n^2}$$

($\xi > 1$)



$$s_{1,2} = -\xi\omega_n \mp \omega_n\sqrt{\xi^2 - 1}$$

($\xi > 1$)

$$s_1 = -\xi\omega_n - \omega_n\sqrt{\xi^2 - 1} \qquad s_2 = -\xi\omega_n + \omega_n\sqrt{\xi^2 - 1}$$